Article ID: 1007- 4627(2000)01-0030-05

Reduction and Self-similarity of Quantum dynamics*

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Abstract: By a generalized version of AGS reduction procedure we show that the forms of quantum dynamics at different strata are the same. This is the self-similarity of quantum dynamics.

Key words: AGS reduction procedure; quantum dynamics; self-similarity
CLC number: O4- 1 **Documber code**: A

1 Introduction

We often need to apply the quantum dynamics at different strata, for examples, apply it to atoms, to the electrons in an atom, to nuclei, to nucleons in a nucleus, and so on. Sometimes we even need to consider a mixed system of objects from different strata. Examples are systems of atoms and electrons, systems of nuclei and nucleons, and systems of hadrons and quarks. We may also need to ignore something and concentrate our attention to other things. For example, in the nuclear reaction theory we would ignore closed channels. In many-body problems, a set of specially chosen particles and elementary excitations is considered instead of original elementary particles composing the system. In all of these cases people always assume that the quantum dynamics takes the same form. However this has not been proven in general from the first principle.

In the following, using the generalized AGS reduction procedure[1~3] we prove the statement of self-similarity for quantum dynamics: The same quantum dynamics may be used to a system of elementary objects as well as to objects, their bound or resonant states, and elementary excitations chosen from this system. We call the system composed of elementary objects the original, call the system composed of chosen objects from the original system the reduced, and call the procedure from the quantum dynamics of original system to the quantum dynamics of reduced system the reduction. The reduction procedure also give a derivation of the Hamiltonian of the reduced system from that of the original system.

In Sec. 2, we review the formal theory for collisions as a fundamental form of quantum dynamics. In Sec. 3, we generalize the AGS reduction procedure and deduce the quantum dynamics for the reduced system by it. Sec. 4 is a discussion.

2 Formal Theory for Collision

Consider the collision processes

$$a + A \to \begin{vmatrix} b + B \\ c + C \\ \cdots \end{vmatrix}$$

in the original system, in which a and A are objects in the ingoing channel, while b and B or c

Received date: 15 August 1999
* **Foundation item**: NSFC (19775004)
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and C and so on are those in outgoing channels. In these channels, objects are considered as to be composed of the elementary objects in the original system. We call s+ S the s channel, either ingoing or outgoing. Define the Hamiltonian of the s channel by

$$H_s = H_0 + H'_s . \tag{1}$$

H_0 is the free Hamiltonian of the system. H'_s is the interaction Hamiltonian in the s channel, including the interaction forming the s and S objects. The interaction distorting the wave function of the relative motion between these objects, or a part of it, may also be included in H'_s. The total Hamiltonian of the system is

$$H = H_s + H_{\bar{s}} . \tag{2}$$

with the definition $H_{\bar{s}} \equiv H - H_s$. The Lippmann-Schwinger equations in the s channel are

$$|sn\pm\rangle = |sn\rangle + \frac{1}{E_{sn} - H_s \pm i\in} H_{\bar{s}} |sn\pm\rangle , \tag{3}$$

in which $|sn\rangle$ is the nth eigenstate of H_s with the eigenenergy E_{sn}, $|sn+\rangle$ and $|sn-\rangle$ are the in-and out-eigenstates of H respectively with the same eigenenergy E_{sn}. The Transition amplitude between states $|an_a\rangle$ and $|bn_b\rangle$ is[1,4]

$$\begin{aligned} &\langle bn_b| U_{ba}(E_{an_a} + i\in)|an_a\rangle \\ &= \langle bn_b| H_{\bar{b}} |an_a +\rangle \\ &= \langle bn_b - |H_{\bar{a}}| an_a\rangle , \end{aligned} \tag{4}$$

with

$$\begin{aligned} U_{ba}(Z) = &-(1-\delta_{ba})(H_0 - Z) + H' - \\ &H'_a - H'_b + \delta_{ab}H'_a - H_{\bar{b}}G(Z)H_{\bar{a}} , \end{aligned} \tag{5}$$

$$\begin{aligned} G(Z) &\equiv (H - Z)^{-1} \\ &= \delta_{ba}G_b(Z) - G_b(Z)U_{ba}(Z)G_a(Z) , \end{aligned} \tag{6}$$

$$G_s(Z) = (H_s - Z)^{-1}, \tag{7}$$

$$H' = H - H_0 . \tag{8}$$

The S-matrix element for the transition is

$$\begin{aligned} \langle bn_b| S|an_a\rangle = &\delta_{ba}\delta_{n_b n_a} - 2\pi i\delta(E_{bn_b} - E_{an_a}) \cdot \\ &\langle bn_b| U_{ba}(E_{an_a} + i\in)|an_a\rangle . \end{aligned} \tag{9}$$

The collision theory is not only for the collision problem, but also for the bound state problem. (5 ~ 9) show that the resolvent G, the transition amplitude U and the scattering matrix S as functions of Z have the same poles, and their real poles $Z = E_n$ form a discrete spectrum of the system, therefore correspond to its bound states. At the pole, the first term on the right hand side of (3) is negligible in comparison with the second term. The solution $|n\rangle$ of the Lippmann-Schwinger equation (3) at the pole E_n is therefore independent of any channel. A stationary state without any incoming and outgoing channel is a bound state. The formal theory for collision is also a form of complete description for quantum dynamics.

3 Generalized AGS Reduction and the Self-similarity of Quantum Dynamics

A channel means a partition of the system. The theory described in the last section is quite general, since there may be several bound parts in s+ S . For an N body s channel with N bound objects $s_1, s_2, \ldots, s_N$, we have s+ S= s_1+ s_2+ ... + s_N. s+ S may also be a bound object, in this case we call it a one-body s channel. Corresponding to the channel s there is also a partition (2) of the Hamiltonian. One may sum up the interaction Hamiltonians H'_s of all channels and define a remainder

$$H'_r \equiv H' - \sum_s H'_s \tag{10}$$

to be the interaction Hamiltonian of a fictitious r channel. The corresponding channel Hamiltonian

$$H_r \equiv H_0 + H_r' \tag{11}$$

may have no any meaningful channel eigenstate. Now let us rewrite(10) in the form

$$H' = \sum_s H_s' , \tag{12}$$

with the understanding that H_r' has been included in the sum on the right. With this relation, we may apply the AGS procedure[1,2] to the system.

Define the T operator of the s channel by

$$T_s(Z) = H_s' - H_s' G_s(Z) H_s' . \tag{13}$$

It is a function of the complex variable Z and satisfies equations

$$\left.\begin{aligned} T_s(Z) G_0(Z) &= H_s' G_s(Z) \\ G_0(Z) T_s(Z) &= G_s(Z) H_s' \end{aligned}\right| , \tag{14}$$

in which

$$G_0(Z) = (H_0 - Z)^{-1} \tag{15}$$

is the free resolvent for the system. Using (6), (7), and (14) we obtain

$$\sum_{b \neq b} T_b' G_0 U_{b'a}' = \sum_{b \neq b} H_b' G_b' U_{b'a}' G_a G_a^{-1}$$
$$= H_a' + H_b' - H' - \delta_{ba} H_b' + H_{\bar{b}} G H_{\bar{a}} . \tag{16}$$

Comparing this equation with(5), we see

$$U_{ba}(Z) = -(1 - \delta_{ba})(H_0 - Z) - \sum_{b \neq b}' T_b'(Z) G_0(Z) U_{b'a}(Z) . \tag{17}$$

Similarly we have

$$U_{ba}(Z) = -(1 - \delta_{ba})(H_0 - Z) - \sum_{a \neq a}' U_{ba'}(Z) G_0(Z) T_a'(Z) . \tag{18}$$

These two equations may be written in the matrix form

$$F = W - WRF \quad \text{and} \quad F = W - FRW \tag{19}$$

respectively, with the matrix elements being defined by

$$F_{ba} = G_0 U_{ba} G_0 , \tag{20}$$

$$W_{ba} = -(1 - \delta_{ba}) G_0 , \tag{21}$$

$$R_{ba} = -\delta_{ba} T_a . \tag{22}$$

Consider the partition

$$T_s = T_s^{(0)} + T_s' , \tag{23}$$

in which

$$T_s^{(0)}(Z) = -\sum_n |sn;\ Z\rangle t_{sn}(Z) \overline{\langle sn;\ Z|} , \tag{24}$$

$$T_s' \equiv T_s - T_s^{(0)} . \tag{25}$$

The partition may be totally arbitrary. In our case we choose the states in its separable part(24) to be

$$|sn;\ Z\rangle = G_0^{-1}(Z) |sn\rangle , \tag{26}$$

$$\overline{\langle sn;\ Z|} = \langle sn| G_0^{-1}(Z) . \tag{27}$$

For a multi-body channel, n is a set of quantum numbers characterizing the inner motion in each bound object and the relative motion between these objects. For an one-body channel, it characterizes the compound state of the system. If the compound state is near a stationary state, and therefore has a long lifetime, we call it a resonant state. Which states should be included in the sum on the right of (24) is determined by the problem under consideration and by the requirement of smallness of the perturbation T_s' to make the solution as easy as possible. Substituting T' for T in(17) and(18) we obtain

$$U_{ba}'(Z) = -(1 - \delta_{ba})(H_0 - Z) - \sum_{b \neq b}' T_b''(Z) G_0(Z) U_{b'a}'(Z) . \tag{28}$$

$$U_{ba}'(Z) = -(1 - \delta_{ba})(H_0 - Z) - \sum_{a \neq a}' U_{ba'}'(Z) G_0(Z) T_a''(Z) . \tag{29}$$

This may be regarded as the definition for U_{ba}'. Likewise, defining

$$F_{ba}' = G_0 U_{ba}' G_0 \quad \text{and} \quad R_{ba}' = -\delta_{ba} T_a' , \tag{30}$$

we may write these equations in the matrix form

$$\left.\begin{aligned} F' &= W - WR'F' \\ F' &= W - F'R'W \end{aligned}\right| . \tag{31}$$

We also have the partition

$$R = R^{(0)} + R' \quad \text{with} \quad R^{(0)}_{ba} = -\delta_{ba} T^{(0)}_a \ . \tag{32}$$

Multiplying two sides of the first equation in(31) by $1- R^{(0)}F$ from right, and defining

$$\tilde{F} = F'(1 - R^{(0)}F) \ , \tag{33}$$

we get

$$\tilde{F} = W - WR^{(0)}F - WR'\tilde{F} \ . \tag{34}$$

It means, if $F= \tilde{F}$, it is if F satisfies

$$F = F' - F'R^{(0)}F \ , \tag{35}$$

the first equation in(19) is satisfied. On the other hand, if that equation is satisfied, then by(34) we have

$$\Delta F = - WR'\Delta F \tag{36}$$

for $\Delta F \equiv F- \tilde{F}$. This is the homogeneous equation of the first linear equation of(31). Suppose that equation is soluble, this equation can have trivial solution $\Delta F = 0$ only. It is $F = \tilde{F}$, and (35) is satisfied. The equivalence of the equation (35) and the first equation in (19) is therefore proven. In the same way, one may prove the equivalence of the equation

$$F = F' - FR^{(0)}F', \tag{37}$$

and the second equation in(19).

A crucial point of the AGS procedure is that the equations (35) and (37) may be closed by the states $|sn\rangle$ chosen in the definition (24) ~ (27) for $T_s^{(0)}$. Defining

$$\begin{aligned} T(Z)_{bn_b, an_a} &\equiv \langle bn_b; Z| F_{ba}| an_a; Z\rangle \\ &= \langle bn_b| U_{ba}(Z)| an_a\rangle \ , \end{aligned} \tag{38}$$

$$\begin{aligned} V(Z)_{bn_b, an_a} &\equiv \langle bn_b; Z| F'_{ba}| an_a; Z\rangle \\ &= \langle bn_b| U'_{ba}(Z)| an_a\rangle \ , \end{aligned} \tag{39}$$

$$G_0(Z)_{bn_b, an_a} \equiv t_{an_a}(Z)\ \delta_{ba}\delta_{n_b n_a} \ , \tag{40}$$

we may write them respectively in the matrix forms

$$T = V - VG_0(Z)\ T \ , \tag{41}$$

$$T = V - TG_0(Z)\ V \ , \tag{42}$$

of the multi-channel Lippmann-Schwinger equation. $G_0(Z)$ is the free resolvent of the equivalent multi-channel problem. Here free means without coupling between channels. The consideration of effects of the channel Hamiltonian H_s for chosen channels has already been completed in G_0. The coupling between channels is included in the effective interaction V. The resultant transition operator is T, it has right relation with the S matrix of the original problem as shown by (38) and (9).

Lippmann-Schwinger equation is equivalent to the Schrödinger equation with suitable boundary condition. The Hamiltonian is defined by

$$H = H_0 + V \ , \tag{43}$$

$$H_0 = G_0^{-1}(Z) + Z \ . \tag{44}$$

Definition(7) shows that the eigenvalue E_{sn} of H_s is a singular point of the function $G_s(Z)$, and therefore, according to(13), is a singular point of $T_s(Z)$. Near the singular point $Z= E_{sn}$,

$$T_s(Z) \approx - H'_s|sn\rangle \frac{1}{E_{sn} - Z} \langle sn| H'_s \ , \tag{45}$$

$$\begin{aligned} H'_s|sn\rangle &= (H_s - H_0)|sn\rangle \\ &= (E_{sn} - H_0)|sn\rangle \\ &\approx (Z - H_0)|sn\rangle \\ &= - G_0^{-1}(Z)|sn\rangle \ . \end{aligned} \tag{46}$$

It shows we may set

$$t_{sn}(Z) = \frac{1}{E_{sn} - Z} \tag{47}$$

in(24). The free Hamiltonian H_0 defined by(44), (40) and (47) is diagonal, and the diagonal elements are eigenenergies E_{sn} of the channels. H_0 operates separately in each channel. In the s channel it reduces to the channel Hamiltonian H_s. For a given energy E, there may be various N-body channel eigenstates with $N = 1, 2, 3, \ldots$ of

the various channel Hamiltonians. Some of the one-body channel states may be identified to be the resonant states. Other channels with $N > 1$ are open channels at this energy. The state vectors are column vectors, each row of them corresponds to an open N-body or a one-body channel. These column vectors open a working state space for the equivalent theory. Their scalar product may be defined as a sum of scalar products between corresponding channel state vectors. Defining the orthogonality of two states by its zero scalar product as usual, states in different channels are then orthogonal to each other in the working space. It means the open channel states are orthogonal to the resonant states in this working space. Since the channel eigenstates form a complete orthonormal set of that channel, the eigenvectors of the free Hamiltonian H_0 form a complete orthonormal set for the working space of the equivalent theory. The quantum dynamics of the original system now is reduced to the quantum dynamics of a system of some chosen objects composed of elementary objects in the original system. The forms of the quantum dynamics for both the original system and the reduced system are the same. It is the self-similarity of the quantum dynamics.

4 Discussion

The above formalism is solely based on the formal quantum theory without any further assumption, and is therefore formally exact and complete. The Hamiltonian is not specified. It may be relativistic or nonrelativistic. For a nuclear system, it may contain or not contain degrees of freedom other than nucleons, may be applied at the hadron level or at the quark-gluon level or on something hybrid. It therefore may, in principle, be applied to the high energy and/or heavy ion nuclear reaction. This has been one of the motivations for developing such a formalism. The forma-lism is developed in the working space. The states in this space may be chosen according to the problem in hand. This makes the formalism highly flexible. One may use this flexibility to join the general theory with various model theories and develop new models in possible new cases.

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量子动力学的约化与自相似*

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摘　要: 推广少体问题中的 AGS 约化理论, 证明量子动力学在不同层次具有相同形式, 称为量子动力学的自相似.

关 键 词: AGS 约化手续; 量子动力学; 自相似

* 基金项目: 国家自然科学基金资助项目(19775004)